Lecture 3 : Martingales: definition, examples

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Wil91, Chapter 10], [Dur10, Section 5.2], [KT75, Section 6.1].

1 Definitions

DEF 3.1 *A* filtered space *is a tuple* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ *where:*

- $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space*
- $\{\mathcal{F}_n\}$ *is a* filtration*, i.e.,*

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_\infty \equiv \sigma(\cup \mathcal{F}_n) \subseteq \mathcal{F}.$$

  where each $\mathcal{F}_i$ *is a* σ*-field.*

Intuitively, $\mathcal{F}_i$ is the information up to time i.

EX 3.2 *Let* $X_0, X_1, \ldots$ *be iid RVs. Then a filtration is given by*

$$\mathcal{F}_n = \sigma(X_0, \ldots, X_n), \ \forall n \geq 0.$$

Fix $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$.

DEF 3.3 *A process* $\{W_n\}_{n \geq 0}$ *is* adapted *if* $W_n \in \mathcal{F}_n$ *for all* n.

Intuitively, the value of W_n is known at time n.

EX 3.4 *Continuing. Let* $\{S_n\}_{n\geq 0}$ *where* $S_n = \sum_{i \leq n} X_i$ *is adapted.*

DEF 3.5 *A process* $\{C_n\}_{n\geq 1}$ *is* previsible *if* $C_n \in \mathcal{F}_{n-1}$ *for all* $n \geq 1$.

EX 3.6 *Continuing.* $C_n = \mathbb{1}\{S_{n-1} \leq k\}$.

Our main definition is the following.

DEF 3.7 *A process* $\{M_n\}_{n\geq 0}$ *is a* martingale *(MG) if*

- $\{M_n\}$ *is adapted*
- $\mathbb{E}|M_n| < +\infty$ *for all* n
- $\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] = M_{n-1}$ *for all* $n \geq 1$

A superMG or subMG is similar except that the equality in the last property is replaced with $\leq$ *or* $\geq$ *respectively.*

2 Examples

EX 3.8 (Sums of iid RVs with mean 0**)** *Let*

- $X_0, X_1, \ldots$ *iid RVs integrable and centered with* $X_0 = 0$
- $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$
- $S_n = \sum_{i \leq n} X_i$

Then note that $\mathbb{E}|S_n| < \infty$ *by the triangle inequality and*

$$\begin{aligned} \mathbb{E}[S_n \,|\, \mathcal{F}_{n-1}] &= \mathbb{E}[S_{n-1} + X_n \,|\, \mathcal{F}_{n-1}] \\ &= S_{n-1} + \mathbb{E}[X_n] = S_{n-1}. \end{aligned}$$

EX 3.9 (Variance of a sum) *Same setup with* $\sigma^2 \equiv \mathrm{Var}[X_1] < \infty$. *Define*

$$M_n = S_n^2 - n\sigma^2.$$

Note that

$$\mathbb{E}|M_n| \leq \sum_{i \leq n} \mathrm{Var}[X_i] + n\sigma^2 \leq 2n\sigma^2 < +\infty$$

and

$$\begin{aligned} \mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] &= \mathbb{E}[(X_n + S_{n-1})^2 - n\sigma^2 \,|\, \mathcal{F}_{n-1}] \\ &= \mathbb{E}[X_n^2 + 2X_n S_{n-1} + S_{n-1}^2 - n\sigma^2 \,|\, \mathcal{F}_{n-1}] \\ &= \sigma^2 + 0 + S_{n-1}^2 - n\sigma^2 = M_{n-1}. \end{aligned}$$

EX 3.10 (Exponential moment of a sum; Wald's MG) *Same setup with* $\phi(\lambda) = \mathbb{E}[\exp(\lambda X_1)] < +\infty$ *for some* $\lambda \neq 0$. *Define*

$$M_n = \phi(\lambda)^{-n} \exp(\lambda S_n).$$

Note that

$$\mathbb{E}|M_n| \leq \frac{\phi(\lambda)^n}{\phi(\lambda)^n} = 1 < +\infty$$

and

$$\begin{aligned} \mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] &= \phi(\lambda)^{-n} \mathbb{E}[\exp(\lambda(X_n + S_{n-1})) \,|\, \mathcal{F}_{n-1}] \\ &= \phi(\lambda)^{-n} \exp(\lambda S_{n-1}) \phi(\lambda) = M_{n-1}. \end{aligned}$$

EX 3.11 (Product of iid RVs with mean 1**)** *Same setup with* $X_0 = 1$*,* $X_i \geq 0$ *and* $\mathbb{E}[X_1] = 1$*. Define*

$$M_n = \prod_{i \leq n} X_i.$$

Note that

$$\mathbb{E}|M_n| = 1$$

and

$$\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] = M_{n-1}\mathbb{E}[X_n \,|\, \mathcal{F}_{n-1}] = M_{n-1}.$$

EX 3.12 (Accumulating data; Doob's MG) *Let* $X \in \mathcal{L}^1(\mathcal{F})$*. Define*

$$M_n = \mathbb{E}[X \,|\, \mathcal{F}_n].$$

Note that

$$\mathbb{E}|M_n| \leq \mathbb{E}|X| < +\infty,$$

and

$$\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] = \mathbb{E}[X \,|\, \mathcal{F}_{n-1}] = M_{n-1},$$

by (TOWER).

EX 3.13 (Eigenvalues of transition matrix) *Recall that a MC on a countable* E *is:*

- $\{\mu_i\}_{i \in E}$, $\{p(i,j)\}_{i,j \in E}$
- $Y(i,n) \sim p(i,\cdot)$ *(indep.)*
- $Z_0 \sim \mu$ *and* $Z_n = Y(Z_{n-1}, n)$*.*

Suppose $f : E \to \mathbb{R}$ *is s.t.*

$$\sum_j p(i,j)f(j) = \lambda f(i),\ \forall i,$$

with $\mathbb{E}|f(Z_n)| < +\infty$ *for all* n*. Define*

$$M_n = \lambda^{-n} f(Z_n).$$

Note that

$$\mathbb{E}|M_n| < +\infty,$$

and

$$\begin{aligned}\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] &= \lambda^{-n}\mathbb{E}[f(Z_n) \,|\, \mathcal{F}_{n-1}] \\ &= \lambda^{-n}\sum_j p(Z_{n-1}, j) f(j) \\ &= \lambda^{-n} \cdot \lambda \cdot f(Z_{n-1}) = M_{n-1}.\end{aligned}$$

EX 3.14 (Branching Process) *Recall that a branching process is:*

- $X(i, n)$*,* $i \geq 1$ *and* $n \geq 1$*, iid with mean* m
- $Z_0 = 1$ *and* $Z_n = \sum_{i \leq Z_{n-1}} X(i, n)$

Note that for $f(j) = j$ *we have*

$$\sum_j p(i, j) j = mi,$$

so that $M_n = m^{-n} Z_n$ *is a MG.*

Further reading

Comments on harmonic functions in [Dur10, Seciton 5.2].

Next class

Stopping times and betting systems [Dur10, Section 5.2].

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[KT75] Samuel Karlin and Howard M. Taylor. *A first course in stochastic processes*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.