Lecture 4 : Martingales: gambling systems

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Wil91, Chapter 10], [Dur10, Section 5.2].

1 Further definition and example

DEF 4.1 *A process* $\{C_n\}_{n\geq 1}$ *is* previsible *if* $C_n \in \mathcal{F}_{n-1}$ *for all* $n \geq 1$.

EX 4.2 *Let* $\{X_n\}_{n\geq 0}$ *be an integrable adapted process and* $\{C_n\}_{n\geq 1}$*, a bounded previsible process. Define*

$$M_n = \sum_{i\leq n}(X_i - \mathbb{E}[X_i \,|\, \mathcal{F}_{i-1}])C_i.$$

Then

$$\mathbb{E}|M_n| \leq \sum_{i\leq n} 2\mathbb{E}|X_n|K < +\infty,$$

where $|C_n| < K$ *for all* $n \geq 1$*, and*

$$\begin{aligned}\mathbb{E}[M_n - M_{n-1} \,|\, \mathcal{F}_{n-1}] &= \mathbb{E}[(X_n - \mathbb{E}[X_n \,|\, \mathcal{F}_{n-1}])C_n \,|\, \mathcal{F}_{n-1}] \\ &= C_n(\mathbb{E}[X_n \,|\, \mathcal{F}_{n-1}] - \mathbb{E}[X_n \,|\, \mathcal{F}_{n-1}]) = 0.\end{aligned}$$

2 Fair games

Take the previous example with $\{X_n\}_{n\geq 0}$ a MG, that is,

$$M_n = (C \bullet X)_n \equiv \sum_{i\leq n} C_i(X_i - X_{i-1}),$$

where $\{(C \bullet X)_n\}_{n\geq 0}$ is called the *martingale transform* and is a discrete analogue of stochastic integration. If you think of $X_n - X_{n-1}$ as your net winnings per unit stake at time n, then C_n is a gambling strategy and $(C \bullet X)$ is your total winnings up to time n in a *fair game*.

Arguing as in the previous example, we have the following theorem.

THM 4.3 (You can't beat the system) *Let* $\{C_n\}$ *be a bounded previsible process and* $\{X_n\}$ *be a MG. Then* $\{(C \bullet X)_n\}$ *is also a MG. If, moreover,* $\{C_n\}$ *is nonnegative and* $\{X_n\}$ *is a superMG, then* $\{(C \bullet X)_n\}$ *is also a superMG.*

3 Stopping times

DEF 4.4 *A random variable* $T : \Omega \to \overline{\mathbb{Z}}_+ \equiv \{0, 1, \ldots, +\infty\}$ *is called a* stopping time *if*

$$\{T \leq n\} \in \mathcal{F}_n, \ \forall n \in \overline{\mathbb{Z}}_+,$$

or, equivalently,

$$\{T = n\} \in \mathcal{F}_n, \ \forall n \in \overline{\mathbb{Z}}_+.$$

In the gambling context, a stopping time is a time at which you decide to stop playing. That decision should only depend on the history up to time n.

EX 4.5 *Let* $\{A_n\}$ *be an adapted process and* $B \in \mathcal{B}$*. Then*

$$T = \inf\{n \geq 0 \, : \, A_n \in B\},$$

is a stopping time.

4 Stopped supermartingales are supermartingales

DEF 4.6 *Let* $\{X_n\}$ *be an adapted process and* T *be a stopping time. Then*

$$X_n^T(\omega) \equiv X_{T(\omega) \wedge n}(\omega),$$

is called $\{X_n\}$ stopped at T.

THM 4.7 *Let* $\{X_n\}$ *be a superMG and* T *be a stopping time. Then the stopped process* X^T *is a superMG and in particular*

$$\mathbb{E}[X_{T \wedge n}] \leq \mathbb{E}[X_0].$$

The same result holds at equality if $\{X_n\}$ *is a MG.*

Proof: Let

$$C_n^{(T)} = \mathbb{1}\{n \leq T\}.$$

Note that

$$\{C_n^{(T)} = 0\} = \{T \leq n - 1\} \in \mathcal{F}_{n-1},$$

so that $C^{(T)}$ is previsible. It is also nonnegative and bounded. Note further that

$$(C^{(T)} \bullet X)_n = X_{T \wedge n} - X_0 = X_n^T - X_0.$$

Apply the previous theorem. ∎

5 Optional stopping theorem

When can we say that $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$?

THM 4.8 *Let $\{X_n\}$ be a superMG and T be a stopping time. Then X_T is integrable and*

$$\mathbb{E}[X_T] \leq \mathbb{E}[X_0].$$

if one of the following holds:

1. *T is bounded*
2. *X is bounded and T is a.s. finite*
3. *$\mathbb{E}[T] < +\infty$ and X has bounded increments*
4. *X is nonnegative and T is a.s. finite.*

The first three hold with equality if X is a MG.

Proof: From the previous theorem, we have

$$(*) \quad \mathbb{E}[X_{T\wedge n} - X_0] \leq 0.$$

1. Take $n = N$ in $(*)$ where $T \leq N$ a.s.
2. Take n to $+\infty$ and use (DOM).
3. Note that
$$|X_{T\wedge n} - X_0| \leq |\sum_{i \leq T\wedge n} (X_i - X_{i-1})| \leq KT,$$
where $|X_n - X_{n-1}| \leq K$ a.s. Use (DOM).
4. Use (FATOU).

■

Further reading

No further reading.

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.