Lecture 5 : Martingale convergence theorem

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Wil91, Chapter 10], [Dur10, Section 5.2].

1 A natural gambling strategy

Recall that

$$(C \bullet X)_n = \sum_{i \leq n} C_n (X_n - X_{n-1}),$$

where C_n is predictable and X_n is a superMG, can be interpreted as your net winnings in a game. A natural strategy is to choose $\alpha < \beta$ and apply the following

- REPEAT
  - Wait until X gets below α
  - Play a unit stake until X gets above β and stop playing
- UNTIL TIME N

More formally, let

$$C_1 = \mathbb{1}\{X_0 < \alpha\},$$

and

$$C_n = \mathbb{1}\{C_{n-1} = 1\}\mathbb{1}\{X_{n-1} \leq \beta\} + \mathbb{1}\{C_{n-1} = 0\}\mathbb{1}\{X_{n-1} < \alpha\}.$$

Then $\{C_n\}$ is predictable.

2 Upcrossings

Define the following stopping times. Let $T_0 = -1$,

$$T_{2k-1} = \inf\{n > T_{2k-2} \,:\, X_n < \alpha\},$$

and

$$T_{2k} = \inf\{n > T_{2k-1} \,:\, X_n > \beta\}.$$

The *number of upcrossings of* $[\alpha, \beta]$ *by time* N is

$$U_N[\alpha, \beta] = \sup\{k \,:\, T_{2k} \leq N\}.$$

LEM 5.1 (Doob's Upcrossing Lemma) *Let* X *be a superMG. Then*

$$(\beta - \alpha)\mathbb{E}U_N[\alpha, \beta] \leq \mathbb{E}[(X_N - \alpha)^-].$$

Proof: Let $Y_n = (C \bullet X)_n$. Then Y_n is a superMG and satisfies

$$Y_N \geq (\beta - \alpha)U_N[\alpha, \beta] - (X_N - \alpha)^-,$$

since $(X_N - \alpha)^-$ overestimates the loss during the last interval of play. The result follows from $\mathbb{E}[Y_N] \leq 0$. ■

COR 5.2 *Let* X *be a superMG bounded in* $\mathcal{L}^1$*. Then*

$$U_N[\alpha, \beta] \uparrow U_\infty[\alpha, \beta],$$

$$(\beta - \alpha)\mathbb{E}U_\infty[\alpha, \beta] \leq |\alpha| + \sup_n \mathbb{E}|X_n| < +\infty,$$

so that

$$\mathbb{P}[U_\infty[\alpha, \beta] = \infty] = 0.$$

Proof: Use (MON). ■

3 Convergence theorem

THM 5.3 (Martingale convergence theorem) *Let* X *be a superMG bounded in* $\mathcal{L}^1$*. Then* X_n *converges and is finite a.s. Moreover, let* $X_\infty = \limsup_n X_n$ *then* $X_\infty \in \mathcal{F}_\infty$ *and* $\mathbb{E}|X_\infty| < +\infty$*.*

Proof: Let $\alpha < \beta \in \mathbb{Q}$ and

$$\Lambda_{\alpha,\beta} = \{\omega \,:\, \liminf X_n < \alpha < \beta < \limsup X_n\}.$$

Note that

$$\begin{aligned}
\Lambda &= \{\omega \,:\, X_n \text{ does not converge}\} \\
&= \{\omega \,:\, \liminf X_n < \limsup X_n\} \\
&= \cup_{\alpha < \beta \in \mathbb{Q}} \Lambda_{\alpha,\beta}.
\end{aligned}$$

Since

$$\Lambda_{\alpha,\beta} \subseteq \{U_\infty[\alpha, \beta] = \infty\},$$

we have $\mathbb{P}[\Lambda_{\alpha,\beta}] = 0$. By countability, $\mathbb{P}[\Lambda] = 0$. Use (FATOU) on $|X_n|$ to conclude. ■

COR 5.4 *If X is a nonnegative superMG then X_n converges a.s.*

Proof: X is bounded in $\mathcal{L}^1$ since

$$\mathbb{E}|X_n| = \mathbb{E}[X_n] \leq \mathbb{E}[X_0],\ \forall n.$$

■

EX 5.5 (Polya's Urn) *An urn contains 1 red ball and 1 green ball. At each time, we pick one ball and put it back with an extra ball of the same color. Let R_n (resp. G_n) be the number of red balls (resp. green balls) after the nth draw. Let $\mathcal{F}_n = \sigma(R_0, G_0, R_1, G_1, \ldots, R_n, G_n)$. Define M_n to be the fraction of green balls. Then*

$$\begin{aligned} \mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] &= \frac{R_{n-1}}{G_{n-1}+R_{n-1}} \frac{G_{n-1}}{G_{n-1}+R_{n-1}+1} \\ &\quad + \frac{G_{n-1}}{G_{n-1}+R_{n-1}} \frac{G_{n-1}+1}{G_{n-1}+R_{n-1}+1} \\ &= \frac{G_{n-1}}{G_{n-1}+R_{n-1}} = M_{n-1}. \end{aligned}$$

Since $M_n \geq 0$ and is a MG, we have $M_n \to M_\infty$ a.s. See [Dur10, Section 4.3] for distribution of the limit and a generalization, or decipher,

$$\mathbb{P}[G_n = m+1] = \binom{n}{m} \frac{m!(n-m)!}{(n+1)!} = \frac{1}{n+1},$$

so that

$$\mathbb{P}[M_n \leq x] = \frac{\lfloor x(n+2) - 1 \rfloor}{n+1} \to x.$$

EX 5.6 (Convergence in L^1?) *We give an example that shows that the conditions of the Martingale Convergence Theorem do not guarantee convergence of expectations. Let $\{S_n\}$ be SRW started at 1 and*

$$T = \inf\{n > 0 \,:\, S_n = 0\}.$$

Then $\{S_{T\wedge n}\}$ is a nonnegative MG. It can only converge to 0 . But $\mathbb{E}[X_0] = 1 \neq 0$.

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.