Lecture 6 : Branching Processes

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References: [Wil91, Chapter 0], [Dur10, Section 4.3], [AN72, Section I.1 - I.5].

1 Branching processes

DEF 6.1 *A* branching process *is an SP of the form:*

- *Let $X(i,n)$, $i \geq 1$, $n \geq 1$, be an array of iid $\mathbb{Z}_+$-valued RVs with finite mean $m = \mathbb{E}[X(1,1)] < +\infty$, and inductively,*

$$Z_n = \sum_{1 \leq i \leq Z_{n-1}} X(i,n)$$

To avoid trivialities we assume $\mathbb{P}[X(1,1) = i] < 1$ for all $i \geq 0$.

LEM 6.2 *$M_n = m^{-n} Z_n$ is a nonnegative MG.*

Proof: Note that we have

$$\sum_j j\mathbb{P}[Z_n = j \,|\, Z_{n-1} = i] = mi,$$

so the claim follows from the eigenvector method. Alternatively, use the following lemma (proved in Hwk 1).

LEM 6.3 *If $Y_1 = Y_2$ a.s. on $B \in \mathcal{F}$ then $\mathbb{E}[Y_1 \,|\, \mathcal{F}] = \mathbb{E}[Y_2 \,|\, \mathcal{F}]$ a.s. on B.*

Then, on $\{Z_{n-1} = k\}$

$$\mathbb{E}[Z_n \,|\, \mathcal{F}_{n-1}] = \mathbb{E}[\sum_{1 \leq j \leq k} X(j,n) \,|\, \mathcal{F}_{n-1}] = mk = mZ_{n-1}.$$

This is true for all k. ■

COR 6.4 *$M_n \to M_\infty < +\infty$ a.s. and $\mathbb{E}[M_\infty] \leq 1$.*

2 Extinction

The martingale convergence theorem in itself tells us little about the limit. Here we try to give a more detailed picture of the limiting behavior—starting with extinction.

Let $p_i = \mathbb{P}[X(1,1) = i]$ for all i and for $s \in [0,1]$

$$f(s) = p_0 + p_1 s + p_2 s^2 + \cdots = \sum_{i \geq 0} p_i s^i.$$

Similarly, $f_n(s) = \mathbb{E}[s^{Z_n}]$. Ideally, we would like to compute the generating function of the limit—but this is rarely possible. Instead, we derive some of its properties. In particular, note that

$$\begin{aligned}
\pi &\equiv \mathbb{P}[Z_n = 0 \text{ for some } n \geq 0] \\
&= \lim_{n \to +\infty} \mathbb{P}[Z_n = 0] \\
&= \lim_{n \to +\infty} f_n(0),
\end{aligned}$$

using the fact that 0 is an absorbing state and monotonicity.

Moreover, by the Markov property, f_n as a natural recursive form:

$$\begin{aligned}
f_n(s) &= \mathbb{E}[s^{Z_n}] \\
&= \mathbb{E}[\mathbb{E}[s^{Z_n} \,|\, \mathcal{F}_{n-1}]] \\
&= \mathbb{E}[f(s)^{Z_{n-1}}] \\
&= f_{n-1}(f(s)) = \cdots = f^{(n)}(s).
\end{aligned}$$

So we need to study iterates of f.

We summarize the properties of f next. To make it easier, we assume $p_0 + p_1 < 1$.

LEM 6.5 *The function f on $[0,1]$ satisfies:*

1. $f(0) = p_0$, $f(1) = 1$
2. *f is indefinitely differentiable on $[0,1)$*
3. *f is strictly convex and increasing*
4. $\lim_{s \uparrow 1} f'(s) = m < +\infty$

Proof: 1. is clear by definition. The function f is a power series with radius of convergence $R \geq 1$. This implies 2. In particular,

$$f'(s) = \sum_{i \geq 1} i p_i s^{i-1} \geq 0,$$

and

$$f''(s) = \sum_{i \geq 2} i(i-1) p_i s^{i-2} > 0.$$

because we must have $p_i > 0$ for some $i > 1$ by assumption. This proves 3. Since $m < +\infty$, $f'(1)$ is well defined and f' is continuous on $[0, 1]$. ∎

COR 6.6 (Fixed points) *We have:*

1. *If $m > 1$ then f has a unique fixed point $\pi_0 \in [0, 1)$*
2. *If $m \leq 1$ then $f(t) > t$ for $t \in [0, 1)$ (Let $\pi_0 = 1$ in that case.)*

Proof: Since $f'(1) = m > 1$, there is $\delta > 0$ s.t. $f(1 - \delta) < 1 - \delta$. On the other hand $f(0) \geq 0$ so by continuity of f there must be a fixed point in $[0, 1 - \delta)$. Moreover, by strict convexity, if r is a fixed point then $f(s) < s$ for $s \in (r, 1)$, proving uniqueness.

The second part follows by strict convexity and monotonicity. ∎

COR 6.7 (Dynamics) *We have:*

1. *If $t \in [0, \pi_0)$, then $f^{(n)}(t) \uparrow \pi_0$*
2. *If $t \in (\pi_0, 1)$ then $f^{(n)}(t) \downarrow \pi_0$*

Proof: We only prove 1. The argument for 2. is similar. By monotonicity, for $t \in [0, \pi_0)$, we have $t < f(t) < f(\pi_0) = \pi_0$. Iterating

$$t < f^{(1)}(t) < \cdots < f^{(n)}(t) < f^{(n)}(\pi_0) = \pi_0.$$

So $f^{(n)}(t) \uparrow L \leq \pi_0$. By continuity of f we can take the limit inside of

$$f^{(n)}(t) = f(f^{(n-1)}(t)),$$

to get $L = f(L)$. So by definition of π_0 we must have $L = \pi_0$. ∎

We immediately obtain:

THM 6.8 (Extinction) *The probability of extinction π is given by the smallest fixed point of f in $[0, 1]$:*

1. *If $m \leq 1$ then $\pi = 1$.*
2. *If $m > 1$ then $\pi < 1$.*

References

[AN72] Krishna B. Athreya and Peter E. Ney. *Branching processes*. Springer-Verlag, New York, 1972.

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.