# Lecture 8 : MGs in $L^p$

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Wil91, Chapter 12], [Dur10, Section 4.4].

## 1 $L^p$ convergence theorem

Recall:

**LEM 8.1 (Markov's inequality)** *Let $Z \geq 0$ be a RV. Then for $c > 0$*

$$c\mathbb{P}[Z \geq c] \leq \mathbb{E}[Z; Z \geq c] \leq \mathbb{E}[Z].$$

MGs provide a useful generalization.

**LEM 8.2 (Doob's submartingale inequality)** *Let $Z \geq 0$ a subMG. Then for $c > 0$*

$$c\mathbb{P}[\sup_{1\leq k\leq n} Z_k \geq c] \leq \mathbb{E}[Z_n; \sup_{1\leq k\leq n} Z_k \geq c] \leq \mathbb{E}[Z_n].$$

**Proof:** Divide $F = \{\sup_{1\leq k\leq n} Z_k \geq c\}$ according to the first time $Z$ crosses $c$:

$$F = F_0 \cup \cdots \cup F_n,$$

where

$$F_k = \{Z_0 < c\} \cap \cdots \cap \{Z_{k-1} < c\} \cap \{Z_k \geq c\}.$$

Since $F_k \in \mathcal{F}_k$ and $\mathbb{E}[Z_n \,|\, \mathcal{F}_k] \geq Z_k$,

$$c\mathbb{P}[F_k] \leq \mathbb{E}[Z_k; F_k] \leq \mathbb{E}[Z_n; F_k].$$

Sum over $k$. ■

**EX 8.3 (Kolmogorov's inequality)** *Let $X_1, \ldots$ be independent RVs with $\mathbb{E}[X_k] = 0$ and $\mathrm{Var}[X_k] < +\infty$. Define $S_n = \sum_{k\leq n} X_k$. Then for $c > 0$*

$$\mathbb{P}[\max_{k\leq n} |S_k| \geq c] \leq c^{-2}\mathrm{Var}[S_n].$$

**THM 8.4 (Doob's $L^p$ inequality)** *Let $p > 1$ and $p^{-1} + q^{-1} = 1$. Let $Z \geq 0$ a subMG bounded in $L^p$. Define*

$$Z^* = \sup_{k \geq 0} Z_k.$$

*Then*

$$\|Z^*\|_p \leq q \sup_k \|Z_k\|_p = q \uparrow \lim_k \|Z_k\|_p.$$

*and $Z^* \in L^p$.*

**Proof:** The last equality follows from (JENSEN). Let $Z_n^* = \sup_{k \leq n} Z_k$. By (MON) it suffices to prove:

**LEM 8.5**

$$\mathbb{E}[(Z_n^*)^p] \leq q^p \mathbb{E}[Z_n^p].$$

**Proof:** Recall the formula: for $Y \geq 0$ and $p > 0$

$$\mathbb{E}[Y^p] = \int_0^\infty p y^{p-1} \mathbb{P}[Y \geq y] dy.$$

Then for $K > 0$

$$\begin{aligned}
\mathbb{E}[(Z_n^* \wedge K)^p] &= \int_0^\infty p c^{p-1} \mathbb{P}[Z_n^* \wedge K \geq c] dc \\
&\leq \int_0^\infty p c^{p-2} \mathbb{E}[Z_n; Z_n^* \wedge K \geq c] dc \\
&= \mathbb{E}\left[Z_n \left(\frac{p}{p-1}\right) \int_0^\infty (p-1) c^{p-2} \mathbb{1}[Z_n^* \wedge K \geq c] dc\right] \\
&= \mathbb{E}[q Z_n (Z_n^* \wedge K)^{p-1}] \\
&\leq q \mathbb{E}[Z_n^p]^{1/p} \mathbb{E}[(Z_n^* \wedge K)^p]^{1/q}.
\end{aligned}$$

Rearranging and using (MON) gives the result. ■ ■

**THM 8.6 ($L^p$ convergence)** *Let $M$ be a MG bounded in $L^p$ for $p > 1$. Then $M_n \to M_\infty$ a.s. and in $L^p$.*

**Proof:** Note that $|M_n|$ is a subMG bounded in $L^p$. In particular, it is bounded in $L^1$ and $M_n \to M_\infty$ a.s. From the previous theorem,

$$|M_n - M_\infty|^p \leq (2 \sup_k |M_k|)^p \in L^1,$$

and by (DOM)

$$\mathbb{E}|M_n - M_\infty|^p \to 0.$$

■

## References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.