Lecture 9 : Martingales in L^2 (continued)

MATH275B - Winter 2012 *Lecturer: Sebastien Roch*

References: [Wil91, Chapter 12], [Dur10, Section 4.4].

1 Review: Random series

Recall:

THM 9.1 (Three-Series Thm) *Let $\{X_n\}$ be independent. For $K > 0$, let $Y_n = X_n \mathbb{1}\{|X_n| \leq K\}$. Then $\sum_n X_n$ converges a.s. if and only if:*

1. $\sum_n \mathbb{P}[|X_n| > K] < +\infty$
2. $\sum_n \mathbb{E}[Y_n]$ *converges*
3. $\sum_n \mathrm{Var}[Y_n] < +\infty$

We will see a MG generalization of this result.

2 Angle-brackets process

THM 9.2 (Doob decomposition) *Let X be an adapted process in L^1. Then*

- *X has an a.s. unique decomposition*

$$X = X_0 + M + A, \qquad (*)$$

where M is a MG and A is predictable with $M_0 = A_0 = 0$.

- *X is a subMG if and only if $A_n \uparrow$ a.s.*

Proof: Suppose $(*)$ holds. Observe

$$\mathbb{E}[X_n - X_{n-1} \,|\, \mathcal{F}_{n-1}] = \mathbb{E}[M_n - M_{n-1} \,|\, \mathcal{F}_{n-1}] + \mathbb{E}[A_n - A_{n-1} \,|\, \mathcal{F}_{n-1}] = A_n - A_{n-1},$$

so that

$$A_n = \sum_{k \leq n} \mathbb{E}[X_k - X_{k-1} \,|\, \mathcal{F}_{k-1}].$$

This proves uniqueness—that is, if there is a decomposition such that M is a MG then A has to be of the previous form. Using this equation as definition gives first claim—by the same equation, M will be a MG. Second claim is now obvious. ■

LEM 9.3 *If M is a MG and ϕ is convex with $\mathbb{E}[|\phi(M_n)|] < +\infty$, then $\phi(M_n)$ is a subMG.*

Proof: Using (cJENSEN)

$$\mathbb{E}[\phi(M_n)\,|\,\mathcal{F}_{n-1}] \geq \phi(\mathbb{E}[M_n\,|\,\mathcal{F}_{n-1}]) = \phi(M_{n-1}).$$

■

DEF 9.4 (Angle-brackets process) *Let M be a MG in $\mathcal{L}^2$ with $M_0 = 0$. Then M^2 is a subMG with decomposition*

$$M^2 \equiv N + \langle M \rangle,$$

where $\langle M \rangle_n \uparrow$ a.s. Moreover M is bounded in L^2 if and only if $\mathbb{E}[\langle M \rangle_\infty] < \infty$. Finally note

$$\langle M \rangle_n = \sum_k \mathbb{E}[M_k^2 - M_{k-1}^2\,|\,\mathcal{F}_{k-1}] = \sum_k \mathbb{E}[(M_k - M_{k-1})^2\,|\,\mathcal{F}_{k-1}].$$

We finally come to our main theorem.

THM 9.5 *Let M be a MG in L^2. Then*

1. $\lim_n M_n(\omega)$ *exists for a.e. ω s.t. $\langle M \rangle_\infty < \infty$.*
2. *If further $|M_n - M_{n-1}| \leq K$ a.s. $\forall n$ then $\langle M \rangle_\infty(\omega) < +\infty$ for a.e. ω s.t.* $\lim_n M_n(\omega)$ *exists.*

Proof: Proof of 1. Observe that

$$\{\langle M \rangle_\infty < \infty\} = \cup_k \{S(k) = +\infty\},$$

where

$$S(k) = \inf\{n \,:\, \langle M \rangle_{n+1} > k\},$$

defines a stopping time. It suffices to prove:

LEM 9.6 $\langle M^{S(k)} \rangle = \langle M \rangle^{S(k)}$.

Indeed, $\mathbb{E}[\langle M\rangle^{S(k)}] \leq k < +\infty$, hence $\mathbb{E}[\langle M^{S(k)}\rangle] < +\infty$ and the MG $M^{S(k)}$ is bounded in L^2:

$$\lim_n M_n^{S(k)} \text{ exists a.s.}$$

Since $S(k) = +\infty$ for some k we have proved the first claim. It remains to prove the lemma. Note that

$$(M^2 - \langle M\rangle)^{S(k)} = (M^{S(k)})^2 - \langle M\rangle^{S(k)},$$

is a MG. By the uniqueness of Doob's decomposition, it suffices to show that $\langle M\rangle^{S(k)}$ is predictable. Let $B \in \mathcal{B}$. Then

$$\{\langle M\rangle_n^{S(k)} \in B\} = E_1 \cup E_2,$$

where

$$E_1 = \cup_{1\leq r\leq n-1}\{S(k) = r,\ \langle M\rangle_r \in B\} \in \mathcal{F}_{n-1},$$

and

$$E_2 = \{S(k) \leq n-1\}^c \cap \{\langle M\rangle_n \in B\} \in \mathcal{F}_{n-1}.$$

That concludes the proof of the first claim.

Proof of 2. (Sketch.) Proof is similar. Enough to prove that $\sup_n |M_n(\omega)| < +\infty$ implies $\langle M\rangle_\infty < +\infty$ a.s. Observe

$$\{\sup_n |M_n(\omega)| < +\infty\} = \cup_c\{T(c) = +\infty\},$$

where

$$T(c) = \inf\{n\ :\ |M_n| > c\},$$

defines a stopping time. By the above lemma,

$$\mathbb{E}[(M_n^{T(c)})^2 - \langle M\rangle_n^{T(c)}] = 0,$$

so that

$$\mathbb{E}[\langle M\rangle_n^{T(c)}] \leq (c+K)^2.$$

Since $T(c) = +\infty$ for some c , this proves the second claim.

∎

3 Applications

THM 9.7 (A strong law for MGs in L^2) *Let M be a MG in $\mathcal{L}^2$ with $M_0 = 0$. Then*

$$\frac{M_n}{\langle M \rangle_n} \to 0, \qquad \text{a.s. on } \{\langle M \rangle_\infty = +\infty\}.$$

Proof: Note that $(1 + \langle M \rangle)^{-1}$ is bounded and predictable so that

$$W_n = ((1 + \langle M \rangle)^{-1} \bullet M)_n = \sum_{k=1}^{n} \frac{M_k - M_{k-1}}{1 + \langle M \rangle_k},$$

is a MG. Note that

$$\begin{aligned}
&\mathbb{E}[(W_n - W_{n-1})^2 \,|\, \mathcal{F}_{n-1}] \\
&\quad = (1 + \langle M \rangle_n)^{-2} \mathbb{E}[(M_n - M_{n-1})^2 \,|\, \mathcal{F}_{n-1}] \\
&\quad = (1 + \langle M \rangle_n)^{-2} (\langle M \rangle_n - \langle M \rangle_{n-1}) \\
&\quad \le (1 + \langle M \rangle_{n-1})^{-1} (1 + \langle M \rangle_n)^{-1} ((1 + \langle M \rangle_n) - (1 + \langle M \rangle_{n-1})) \\
&\quad = (1 + \langle M \rangle_{n-1})^{-1} - (1 + \langle M \rangle_n)^{-1}.
\end{aligned}$$

In particular, $\langle W \rangle_\infty \le 1 < +\infty$ so that W_n converges a.s.

LEM 9.8 (Kronecker's Lemma) *If $b_n \uparrow +\infty$ then*

$$\sum_n \frac{x_n}{b_n} \text{ converges} \qquad \Longrightarrow \qquad \frac{\sum_n x_n}{b_n} \to 0.$$

Then on $\{\langle M \rangle_\infty = +\infty\}$, we have $M_n/(1 + \langle M \rangle_n) \to 0$ and the result follows. ■

THM 9.9 (Levy's extension of Borel-Cantelli) *Suppose $\mathbb{1}_{E_k}$ is adapted. Define*

$$Z_n = \sum_{k=1}^{n} \mathbb{1}_{E_k},$$

and

$$Y_n = \sum_{k=1}^{n} \mathbb{P}[E_k \,|\, \mathcal{F}_{k-1}].$$

Then

1. $Y_\infty < \infty \implies Z_\infty < \infty$
2. $Y_\infty = +\infty \implies Z_n/Y_n \to 1$

Note that the previous theorem implies the classical BC lemmas. For 1, note that $\mathbb{E}[Y_\infty] = \sum_k \mathbb{P}[E_k]$. For 2, note that by independence $\mathbb{P}[E_k \,|\, \mathcal{F}_{k-1}] = \mathbb{P}[E_k]$.
Proof: Z is a subMG, Y is predictable and $M = Z - Y$ is a MG. The proof relies on computing $\langle M \rangle$. Note

$$\begin{aligned}
\langle M \rangle_n &= \sum_{k=1}^n \mathbb{E}[(M_k - M_{k-1})^2 \,|\, \mathcal{F}_{k-1}] \\
&= \sum_{k=1}^n \mathbb{E}[(\mathbb{1}_{E_k} - \mathbb{P}[E_k \,|\, \mathcal{F}_{k-1}])^2 \,|\, \mathcal{F}_{k-1}] \\
&= \sum_{k=1}^n \mathbb{E}[\mathbb{1}_{E_k} - \mathbb{P}[E_k \,|\, \mathcal{F}_{k-1}]^2 \,|\, \mathcal{F}_{k-1}] \\
&= \sum_{k=1}^n [\mathbb{P}[E_k \,|\, \mathcal{F}_{k-1}] - \mathbb{P}[E_k \,|\, \mathcal{F}_{k-1}]^2] \\
&\leq Y_n.
\end{aligned}$$

We are ready to prove the statements.

1. $\boxed{Y_\infty < +\infty.}$ Then $\langle M \rangle_\infty < +\infty$ and M_n converges. Hence, $Z = M + Y$ also converges.

2. $\boxed{Y_\infty = +\infty.}$ Assume first that $\langle M \rangle_\infty < +\infty$. Then M_n converges and

$$\frac{Z_n}{Y_n} = \frac{M_n + Y_n}{Y_n} \to 1.$$

On the other hand, if $\langle M \rangle_\infty = +\infty$ the strong law for L^2 MGs gives $M_n / \langle M \rangle_n \to 0$ so that $M_n / Y_n \to 0$ and $Z_n / Y_n \to 1$.

■

References

[Dur10] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.